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On the geometry of chiral dynamics: II

J. S. DOWKER

Department of Theoretical Physics, University of Manchester

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Abstract. Previous work on the geometry of group space appropriate to the problem of chiral symmetry is extended to cover 'other fields' with the object of elucidating the relation between those fields which transform linearly and those which do not. The different fields just correspond to different choices of the local coordinates which must be set up to define 'spinors'.

The results for specific quantities such as transformation matrices and covariant derivatives agree with those of other papers.

1. Introduction

This paper is essentially a continuation of a previous one (Dowker 1970, to be referred to as I) and, to save repetition, we refer to this last for all undefined quantities. In I the basic structure of group space was detailed† and the chiral group $G \otimes G$ introduced as the (connected) group of motions on the group space of G. The 0^- mesons are associated with the coordinates of this space.

2. The 'other fields' problem

We now must introduce nucleons (for SU(2)) and quarks (for other groups) into the scheme. Meetz (1969) has already employed a method of Pauli's for the introduction of spin into general relativity (i.e. into curved space–time) in this connection (see e.g. De Witt and De Witt 1952). In some ways this is confusing since the situation is really more akin to that in flat space–time, the essential thing being the existence of groups of motions in the space. Thus in a general space–time it is not possible to introduce a spinor as a geometrical object (Cartan 1938). One can only introduce them locally through the *orthogonal* transformations (homogeneous groups of motions) of the flat tangent spaces (see, for example, Dowker and Dowker 1966 for a general discussion and literature). Under general coordinate transformations the spinor fields transform as scalars. That is, the spinor space transformations are quite separate from the general coordinate transformations of the underlying manifold. The link between these two spaces is provided by the connecting quantities like the Pauli or Dirac matrices which transform like vectors under general coordinate transformations. This means, for example, that the bilinear object $\bar{\psi}\gamma^\mu\psi$ is a vector field in the curved space. If the flat space limit of this situation is taken, then all the tangent spaces merge with the underlying manifold essentially into one flat space. We are now in the realm of special relativity and Lorentz transformations and we see that under these latter the spinor ψ does not transform but that the γ^μ do. The important thing is that the physical bilinear $\bar{\psi}\gamma^\mu\psi$ should be a vector. Normally one takes the γ^μ as purely numerical quantities not affected by Lorentz transformations and the field then carries the full transformation. This is possible because of the theorem that if $x^\mu \rightarrow a^\mu_{.\nu}x^\nu$ is a Lorentz transformation then there exists a non-singular matrix S such that

$$S\gamma^\mu S^{-1} = a^\mu_{.\nu}\gamma^\nu.$$

† In addition to the references in I the reader will find in Hodge (1952, chap. V) a rapid survey of the geometry of group space together with some interesting topological results.

S is now the spinor transformation matrix. We see that the essential thing is the existence of a set of preferred coordinate frames (Cartesian) which are transformed into each other by a group of motions (Lorentz transformations). All this is well known.

The situation is entirely analogous in group space, the role of the Lorentz transformations being played by K. Preferred coordinate systems will be preferred anholonomic, or local, coordinates defined by fields A^{α}_{a}, A^{α}_{A} satisfying at each point the condition

$$c_{ab}^{\;\;c} = A^{\alpha}_{a}A^{\beta}_{b}A^{c}_{\gamma}c_{\alpha\beta}^{\;\;..\gamma}, \qquad c_{AB}^{\;\;C} = A^{\alpha}_{A}A^{\beta}_{B}A^{C}_{\gamma}c_{\alpha\beta}^{\;\;..\gamma}$$

$$g_{ab} = A^{\alpha}_{a}A^{\beta}_{b}g_{\alpha\beta}, \qquad g_{AB} = A^{\alpha}_{A}A^{\beta}_{B}g_{\alpha\beta}.$$

Given such a set at one point if it is 'dragged along' by the point transformation $\xi \to {}'\xi$ belonging to K, it becomes another such set. This follows from the equations

$$\left.\begin{aligned} &\underset{A}{\pounds}A^{\alpha}_{a} = 0 = \underset{a}{\pounds}A^{\alpha}_{A} \\ &\underset{a}{\pounds}A^{\alpha}_{b} = c_{ab}^{\;\;c}A^{\alpha}_{c}; \qquad \underset{A}{\pounds}A^{\alpha}_{B} = -\,c_{AB}^{\;\;C}A^{\alpha}_{C} \end{aligned}\right\} \tag{1}$$

which can be shown analytically, using, for example, the equivalence of 'dragging along' and parallel displacement given in I, § 3.

Thus, under 'dragging along' by a point transformation belonging to the first (second) parameter group, the $A^{\alpha}_{a}(A^{\alpha}_{A})$ suffer a transformation belonging to the adjoint group while the $A^{\alpha}_{A}(A^{\alpha}_{a})$ remain invariant. In other words if we define 'dragged along' fields, $\overset{m}{A}{}^{\alpha}_{a}$, $\overset{m}{A}{}^{\alpha}_{A}$, we have, at every point,

$$\left.\begin{aligned} \overset{m}{A}{}^{\alpha}_{a} &= A^{\alpha}_{a} \\ \overset{m}{A}{}^{\alpha}_{A} &= D^{B}_{A}A^{\alpha}_{B} \end{aligned}\right\} \text{ under second parameter group}$$

$$\left.\begin{aligned} \overset{m}{A}{}^{\alpha}_{A} &= A^{\alpha}_{A} \\ \overset{m}{A}{}^{\alpha}_{a} &= D^{b}_{a}A^{\alpha}_{b} \end{aligned}\right\} \text{ under first parameter group .}$$

Thus the analogue of the Lorentz group is the adjoint group and so we introduce the linear representations of the latter and denote the corresponding vectors $\overset{\mathrm{L}}{\phi}$, $\overset{\mathrm{R}}{\phi}$. Under left (right) transformations $\overset{\mathrm{L}}{\phi}(\overset{\mathrm{R}}{\phi})$ undergoes the adjoint transformation while $\overset{\mathrm{R}}{\phi}(\overset{\mathrm{L}}{\phi})$ remains invariant. Under reflection $\overset{\mathrm{L}}{\phi}$ and $\overset{\mathrm{R}}{\phi}$ exchange places.

In symbols, under a transformation of K, $\xi \to {}'\xi = \eta_{\mathrm{L}}\xi\eta_{\mathrm{R}}$ we have

$$\overset{\mathrm{L}}{\phi} \to {}'\overset{\mathrm{L}}{\phi} = \exp(\mathrm{i}\epsilon^{a}J_{a})\overset{\mathrm{L}}{\phi}$$

$$\overset{\mathrm{R}}{\phi} \to {}'\overset{\mathrm{R}}{\phi} = \exp(\mathrm{i}\epsilon^{A}J_{A})\overset{\mathrm{R}}{\phi}$$

where the left and right 'spinor' parameters, ϵ^{a} and ϵ^{A}, depend on only the parameters of η_{L} and η_{R} respectively. The J_{a} and J_{A} are the matrix generators of the left and right adjoint groups in an arbitrary representation. For the adjoint representations we have specifically

$$(J_a)^c_b = \mathrm{i}c_{ab}^{\ \ c}$$

$$(J_A)^C_B = -\mathrm{i}c_{AB}^{\ \ C}$$

and, in view of the numerical equality of the $c_{ab}^{\ \ c}$ and $c_{AB}^{\ \ C}$ we can put in general

$$J_a = -\delta^A_a J_A.$$

This is, of course, only a numerical equality. J_a and J_A are matrix operators in *different* vector spaces. It is possible to carry the calculation through for an arbitrary representation but for the moment we shall confine ourselves to the simplest one, i.e. the fundamental or 'quark' representation, and define the corresponding generators λ_a, λ_A which can be defined by the multiplication law

$$\lambda_a\lambda_b = \frac{2}{n}g_{ab}+(d_{abc}+\mathrm{i}c_{ab}^{\ \ c})\lambda_c \tag{2}$$

for SU(n), where

$$d_{abc} = \tfrac{1}{2}\,\mathrm{Tr}[\lambda_{(a}\lambda_b\lambda_{c)}].$$

Let us just consider the SU(n) case from now on. The λ_A satisfy the same equation as the λ_a except that the λ term on the right-hand side is reversed in sign.†

We now wish to investigate more closely the relation of the ϵ^a and ϵ^A to the parameters of η_{L} and η_{R}. To do this we define linking quantities between the adjoint spaces and the underlying manifold X_r, i.e. we construct bilinear functions of $\overset{\mathrm{L}}{\phi}$ and $\overset{\mathrm{R}}{\phi}$ that transform as tensor fields in X_r. We do not wish to go into details of the general theory of these objects but shall simply state the almost obvious that, for the quark representation, the required quantities are the generalizations of the λ_a to the curved space. Thus we define $\overset{\mathrm{L}}{\lambda}_\alpha$ and $\overset{\mathrm{R}}{\lambda}_\alpha$ by

$$\overset{\mathrm{L}}{\lambda}_\alpha\overset{\mathrm{L}}{\lambda}_\beta = \frac{2}{n}g_{\alpha\beta}+(d_{\alpha\beta\gamma}+\mathrm{i}c_{\alpha\beta\gamma})\overset{\mathrm{L}}{\lambda}{}^\gamma \tag{3L}$$

$$\overset{\mathrm{R}}{\lambda}_\alpha\overset{\mathrm{R}}{\lambda}_\beta = \frac{2}{n}g_{\alpha\beta}-(d_{\alpha\beta\gamma}+\mathrm{i}c_{\alpha\beta\gamma})\overset{\mathrm{R}}{\lambda}{}^\gamma \tag{3R}$$

where $d_{\alpha\beta\gamma}$ is given by

$$d_{\alpha\beta\gamma} = A^a_\alpha B^b_\beta A^c_\gamma d_{abc} = A^A_\alpha A^B_\beta A^C_\gamma d_{ABC}.$$

The $\overset{\mathrm{L}}{\lambda}_\alpha$ and $\overset{\mathrm{R}}{\lambda}_\alpha$ are, of course, functions of ξ.

Earlier we had espoused the view that, under K, the fields $\overset{\mathrm{L}}{\phi}$ and $\overset{\mathrm{R}}{\phi}$ transformed according to the adjoint group. In this case the λ_α will be numerically invariant in the sense that they will transform as‡

$$\lambda_\alpha(\xi) \to \lambda_\alpha(\xi').$$

This corresponds to the usual attitude in special relativity. However, as was discussed previously, we can just as well say that the ϕ transform as scalars,

$$\phi(\xi) \to \phi'(\xi') = \phi(\xi)$$

and that the λ_α transform as

$$\lambda_\alpha(\xi) \to S^{-1}\lambda_\alpha(\xi')S.$$

† $d_{ABC} \equiv -\frac{1}{2}\mathrm{Tr}[\lambda_{(A}\lambda_B\lambda_{C)}] = \delta^a_A\delta^b_B\delta^c_C\, d_{abc}$. See the interesting and relevant discussion by Biedenharn (1963) on the invariants of SU(n).

‡ Where convenient we leave off the symbols L and R.

The coordinate transformation here is that *induced* by the point transformation $\xi \to {}'\xi$, i.e. $\xi^{\alpha'} = {}'\xi^{\alpha}$. It is the *inverse* of the dragged along transformation (which we could also have used). S is an adjoint quark representation matrix.

The second attitude is, in many ways, a more useful one. It allows us to discuss directly general coordinate transformations and not just those induced by K†. Under general coordinate transformations the ϕ are scalars and the λ^{α} transform as vectors,

$$\lambda^{\alpha}(\xi) \to \lambda^{\alpha'}(\xi') = \frac{\partial \xi^{\alpha'}}{\partial \xi^{\beta}} \lambda^{\beta}(\xi) \tag{4}$$

so that the bilinears

$$\overset{\mathrm{L}}{\phi}{}^{\dagger} \overset{\mathrm{L}}{\lambda}{}^{\alpha} \overset{\mathrm{L}}{\phi}, \qquad \overset{\mathrm{R}}{\phi}{}^{\dagger} \overset{\mathrm{R}}{\lambda}{}^{\alpha} \overset{\mathrm{R}}{\phi}$$

form vector fields. When the general coordinate transformation happens to be one that can be induced by a member of K the $\lambda^{\alpha'}$ satisfy the same multiplication law as the λ^{α}, i.e. substituting $\delta^{\alpha}_{\alpha'} \lambda^{\alpha'}$ for λ^{α} in (3) yields equations that are still true. We can now use the theorem that two solutions of (3) are related by

$$\delta^{\alpha}_{\alpha'} \lambda^{\alpha'} = S^{-1} \lambda^{\alpha} S \tag{5}$$

where S is an adjoint matrix, as before.

Let us assume that the transformation $\xi \to {}'\xi$ belonging to K is an infinitesimal one,

$$\xi^{\alpha} \to \xi^{\alpha} + \mathrm{d}\xi^{\alpha} = \xi^{\alpha} + e^{a}\, \mathrm{d}t\, A^{\alpha}_{a}(\xi) + e^{A}\, \mathrm{d}t\, A^{\alpha}_{A}(\xi).$$

$$(e^{a},\ e^{A} = \text{constants})$$

Then we shall write S as

$$S = 1 + \mathrm{i} T\, \mathrm{d}t$$

where T is given separately for left and right transformations by

$$\overset{\mathrm{L}}{T} = \tfrac{1}{2} \epsilon^{a} \lambda_{a}, \qquad \overset{\mathrm{R}}{T} = \tfrac{1}{2} \epsilon^{A} \lambda_{A}$$

and we have conveniently separated out a factor of $\mathrm{d}t$ from the ϵ^{a} and ϵ^{A}. The problem now is to determine the relation between the ϵ^{a}, ϵ^{A} and the e^{a}, e^{A}. This easily follows from (4) and (5). Since the λ^{α} is a vector field then its Lie derivative over the general transformation $\xi^{\alpha} \to {}'\xi^{\alpha} = \xi^{\alpha} + v^{\alpha}\, \mathrm{d}t$ is (e.g. Schouten 1954, p. 105)

$$\underset{v}{\pounds} \lambda^{\alpha} = v^{\beta} \partial_{\beta} \lambda^{\alpha} - \lambda^{\beta} \partial_{\beta} v^{\alpha}.$$

(This is just a fancy way of writing (4).) In the case where this transformation belongs to K, (5) implies that

$$\underset{v}{\pounds} \lambda^{\alpha} = \mathrm{i}[T, \lambda^{\alpha}].$$

Hence we get the condition

$$\mathrm{i}[T, \lambda^{\alpha}] = \underset{v}{\pounds} \lambda^{\alpha} \equiv v^{\beta} \partial_{\beta} \lambda^{\alpha} - \lambda^{\beta} \partial_{\beta} v^{\alpha} = -\underset{\lambda}{\pounds} v^{\alpha} \tag{6}$$

where

$$v^{\alpha} = e^{a} A^{\alpha}_{a} + e^{A} A^{\alpha}_{A} \equiv \overset{\mathrm{L}}{e}{}^{\alpha} + \overset{\mathrm{R}}{e}{}^{\alpha}. \tag{7}$$

If we have that the $\overset{\mathrm{L}}{\phi}$ and $\overset{\mathrm{R}}{\phi}$ are invariant‡ under the right and left groups respectively

† That is we wish to include allowable coordinates rather than just preferred ones.

‡ Note that we are using the usual language here and we shall occasionally revert to this usage from force of habit.

then the corresponding T must vanish. This implies that the λ^α must be linear combinations of the A^α. Thus we have in particular

$$\overset{\mathrm{L}}{\lambda}{}^\alpha = A^\alpha_a\lambda^a, \qquad \overset{\mathrm{R}}{\lambda}{}^\alpha = A^\alpha_A\lambda^A$$

which are consistent with (2) and (3), and we can write

$$T = \tfrac{1}{2}\epsilon^\alpha\lambda_\alpha. \tag{8}$$

If we substitute (7) and (8) into (6) we easily find that

and so
$$\left.\begin{aligned} &\epsilon^\alpha = e^\alpha \\ &\epsilon^a = e^a, \qquad \epsilon^A = e^A \end{aligned}\right\}. \tag{9}$$

(The formulae

$$\begin{aligned} \underset{v}{\pounds} u^\alpha &= v^\beta \overset{+}{\nabla}_\beta u^\alpha - u^\beta \overset{-}{\nabla}_\beta v^\alpha \\ &= v^\beta \overset{-}{\nabla}_\beta u^\alpha - u^\beta \overset{+}{\nabla}_\beta v^\alpha = v^\beta \nabla_\beta u^\alpha - u^\beta \nabla_\beta v^\alpha \end{aligned}$$

are sometimes useful.)

Equation (9) is what we expected because the left and right groups have the same structure. It also shows that the ϵ^a and ϵ^A are constants. Further, under 'isospin' transformations $\overset{\mathrm{L}}{\phi}$ and $\overset{\mathrm{R}}{\phi}$ transform in the same way, while under chiral transformations they transform oppositely. Thus the $\overset{\mathrm{L}}{\phi}$ and $\overset{\mathrm{R}}{\phi}$ are to be identified with the quark fields *before* the Gürsey transformation† (Gürsey 1960, Chang and Gürsey 1967, Cronin 1967, Brown 1967). We must, however, be a little more careful about Lorentz transformations when introducing the quarks. Let us only consider massive spin-half particles for which we 'must' have a Dirac spinor ψ, split into upper and lower two-spinors φ and χ, which are also iso-spinors,

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}.$$

Since φ and χ exchange places under reflection (we adopt, temporarily, the usual way of speaking) and so do left and right fields, we see that we have two possibilities, corresponding to left and right fields for both φ and χ

$$\psi = \begin{pmatrix} \overset{\mathrm{L}}{\varphi} \\ \\ \overset{\mathrm{R}}{\chi} \end{pmatrix} \qquad \text{and} \qquad \psi = \begin{pmatrix} \overset{\mathrm{R}}{\varphi} \\ \\ \overset{\mathrm{L}}{\chi} \end{pmatrix}. \tag{10}$$

If the first of these is identified with the fundamental baryons the second will be their chiral partners.

The mass term in the quark Lagrangian is $\kappa\bar{\psi}\psi$ which, for our γ-matrix representation, mixes upper and lower components

$$\kappa\bar{\psi}\psi = \mathrm{i}\kappa(\chi^\dagger\phi - \phi^\dagger\chi)$$

i.e., according to (10), mixes left and right fields. Thus, because of Lorentz invariance it is not possible to get away with chiral invariants constructed from $\overset{\mathrm{L}}{\phi}$ or $\overset{\mathrm{R}}{\phi}$ separately such as $\overset{\mathrm{L}}{\phi}{}^\dagger\overset{\mathrm{L}}{\phi}$ or $\overset{\mathrm{R}}{\phi}{}^\dagger\overset{\mathrm{R}}{\phi}$. We need invariants containing both $\overset{\mathrm{L}}{\varphi}$ and $\overset{\mathrm{R}}{\chi}$.

† cf. also Weinberg (1968), § V.

To construct these we make use of the important fact that the (a) and (A) frames are related at each point by an adjoint representation transformation. Specifically this relation is (e.g. Schouten 1954, p. 200),

$$A^a_\alpha(\xi) = D^a_b(\xi)\delta^b_B A^B_\alpha(\xi) \equiv D^a_B(\xi)A^B_\alpha(\xi) \tag{11}$$

and can be proved directly or from the explicit forms of the A^α_a and A^α_A in canonical coordinates given in I (14). In canonical coordinates (i) $D^a_b(\xi)$ takes the form

$$D^a_b(\xi) = [\exp(-U)]^a_b$$

where

$$U = U(\xi) = \xi^a C_a, \qquad [C_a]^c_b = c_{ab}^{\ \ c}.$$

It is now reasonably clear what is going to happen. To spell out the details we note the relation between $\overset{L}{\lambda}$ and $\overset{R}{\lambda}$ following from

$$\overset{L}{\lambda}_\alpha = A^a_\alpha \lambda_a = A^B_\alpha D^a_B \lambda_a.$$

Thus, by the fundamental theorem, we can write

$$\overset{L}{\lambda}_\alpha = -W^{-1}\overset{R}{\lambda}_\alpha W, \qquad W = \exp(\mathrm{i}\tfrac{1}{2}\omega^a\lambda_a), \qquad \omega^a = \omega^a(\xi)$$

and so

$$\overset{L}{\phi}{}^\dagger \overset{L}{\lambda}_\alpha \overset{L}{\phi} = -\overset{L}{\phi}{}^\dagger W^{-1}\overset{R}{\lambda}_\alpha W\overset{L}{\phi}$$

which shows that $W\overset{L}{\phi}$ transforms like $\overset{R}{\phi}$ and invariants

$$\overset{R}{\phi}{}^\dagger W\overset{L}{\phi}, \qquad \overset{L}{\phi}{}^\dagger W^{-1}\overset{R}{\phi}$$

can be constructed for general left and right fields. If we use canonical coordinates the parameters ω^a in the 'metric' W are just the coordinates of the point we are at, i.e.

$$\omega^a = -\xi^a.$$

The relation of this analysis with that of Gürsey (1960) and Chang and Gürsey (1967) is now apparent. The different forms of the transformation matrix discussed in the latter paper correspond to different coordinatizations of group space, as is well known. The exponential form is singled out as that resulting from using canonical coordinates and, therefore, must have a somewhat privileged position.

We may now perform the standard unitary transformation to new fields $\overset{L}{\phi}_0$ and $\overset{R}{\phi}_0$ where

$$\overset{L}{\phi}_0 = W^{1/2}\overset{L}{\phi}, \qquad \overset{R}{\phi}_0 = W^{-1/2}\overset{R}{\phi}.$$

The question now arises as to whether these fields can be introduced more directly. Let us go back to where we introduced spinor fields for the first time. The fields $\overset{L}{\phi}$ and $\overset{R}{\phi}$ emerge most naturally but we can give a more general approach in the following way. Firstly we define quantities λ^α satisfying (3L). Equations (4) and (5) are valid where

$$S = \exp(\mathrm{i}\tfrac{1}{2}\epsilon^\alpha\lambda_\alpha).$$

Scalar fields ϕ are now introduced to make the bilinear quantity $\phi^\dagger\lambda^\alpha\phi$ a vector field. We now say that the most general solution of (3L) is

$$\lambda^\alpha = A^\alpha_a S^a_b \lambda^b$$

where λ^b satisfy (2) and have *fixed* constant values. $\bar{S}^a_b$ is an adjoint representation matrix which is, in general, a function of position

$$\bar{S} = \exp(\bar{\sigma}^a C_a), \qquad \bar{\sigma}^a = \bar{\sigma}^a(\xi).$$

The different types of fields now result from the different choices of $\bar{S}$. Thus we have the two types $\overset{\mathrm{L}}{\phi}$ and $\overset{\mathrm{L}}{\phi}_0$ for which $\bar{S}$ is 1 and $\exp(\frac{1}{2}\xi^a C_a)$ respectively, if we use canonical coordinates.

Similar considerations apply to the 'right' quantities. Thus the most general solution of (3R) is

$$\lambda^\alpha = A^\alpha_A \bar{R}^A_B \lambda^B$$

where

$$\bar{R}^A_B = \exp(\bar{\rho}^a C_a)^a_b \delta^A_a \delta^b_B, \qquad \bar{\rho}^a = \bar{\rho}^a(\xi)$$

and we obtain the two types $\overset{\mathrm{R}}{\phi}$ and $\overset{\mathrm{R}}{\phi}_0$ if $\bar{\rho}^a$ equals 0 and $-\frac{1}{2}\xi^a$ respectively.

If we like we can start from, say, just the left quantities and obtain all others through adjoint rotations and reflections.

What we are saying is that to introduce spinors into X_r we need a set of preferred local coordinates. The structure of the space throws up two such sets, A^α_a and A^α_A, and hence the spinors $\overset{\mathrm{L}}{\phi}$ and $\overset{\mathrm{R}}{\phi}$. But we are not restricted to just these two sets. As we have said there is no *a priori* or necessary connection between spinor space and X_r.

The ϕ_0 fields have the important property that $\overset{\mathrm{L}}{\phi}_0$ transforms like $\overset{\mathrm{R}}{\phi}_0$. This follows from the relation (11) which can be rewritten

$$[\exp(\tfrac{1}{2}U)]^c_a A^a_\alpha = [\exp(-\tfrac{1}{2}U)]^c_b \delta^b_B A^B_\alpha.$$

We thus reach the same conclusions as before and can put, for the 'nucleon' and its chiral partner,

$$\psi_0 = \begin{pmatrix} \overset{\mathrm{L}}{\varphi}_0 \\ \\ \overset{\mathrm{R}}{\chi}_0 \end{pmatrix}, \qquad \psi_0 = \begin{pmatrix} \overset{\mathrm{R}}{\varphi}_0 \\ \\ \overset{\mathrm{L}}{\chi}_0 \end{pmatrix}.$$

The mass term $\bar{\psi}_0\psi_0$ is then chiral invariant.

We now wish to determine how the ϕ_0 fields transform under K†. It is sufficient to discuss just the left fields and we look at the quantities

$$\overset{\mathrm{L}}{\underset{\circ}{\lambda}}_\alpha(\xi) = \lambda_d[\exp(\tfrac{1}{2}\xi^a C_a)]^d_{.b} A^b_\alpha(\xi).$$

If $\xi \to {}'\xi$ is a motion we expect these to transform as

$$\overset{\mathrm{L}}{\underset{\circ}{\lambda}}_\alpha(\xi) \to S^{-1}\overset{\mathrm{L}}{\underset{\circ}{\lambda}}_\alpha({}'\xi)S$$

and we wish to find $S = 1+\mathrm{i}T$.

The first step is the transformation

$$\overset{\mathrm{L}}{\underset{\circ}{\lambda}}_\alpha(\xi) \to \lambda_d[\exp(\tfrac{1}{2}\xi^a C_a)]^d_{.b}\, {}'A^b_\alpha({}'\xi).$$

(Note that the factor $\exp(\frac{1}{2}\xi^a C_a)$ transforms as a scalar.) Under the transformation

$$\xi \to {}'\xi = \eta_{\mathrm{L}}\xi\eta_{\mathrm{R}}$$

† We could use equation (6) but we prefer to repeat the analysis.

belonging to K, it is easy to show that the A^a_α change as

$$'A^b_\alpha('\xi) = [\exp(\eta^a_{\mathrm{L}} C_a)]^b_{.c} A^c_\alpha('\xi)$$

(cf. equation (1)).

Thus

$$\overset{\mathrm{L}}{\underset{\circ}{\lambda}}_\alpha(\xi) \to \lambda_d[\exp(\tfrac{1}{2}\xi^a C_a)]^d_{.b}[\exp(\eta^a_{\mathrm{L}} C_a)]^b_{.c} A^c_\alpha('\xi).$$

If we can find parameters σ^a such that†

$$[\exp(\tfrac{1}{2}\xi^a C_a)]^d_{.b}[\exp(\eta^a_{\mathrm{L}} C_a)]^b_{.c} = [\exp(\sigma^a C_a)]^d_{.b}[\exp(\tfrac{1}{2}'\xi^a C_a)]^b_{.c} \qquad (12)$$

we can determine S because we would then have

$$\begin{aligned}\overset{\mathrm{L}}{\underset{\circ}{\lambda}}_\alpha(\xi) &\to \lambda_d[\exp(\sigma^a C_a)]^d_{.b}[\exp(\tfrac{1}{2}'\xi^a C_a)]^b_{.c} A^c_\alpha('\xi)\\ &= \exp(-\tfrac{1}{2}\mathrm{i}\sigma^a\lambda_a)\lambda_b \exp(\tfrac{1}{2}\mathrm{i}\sigma^a\lambda_a)[\exp(\tfrac{1}{2}'\xi^a C_a)]^b_{.c} A^c_\alpha('\xi)\\ &= \exp(-\tfrac{1}{2}\mathrm{i}\sigma^a\lambda_a)\overset{\mathrm{L}}{\underset{\circ}{\lambda}}_\alpha('\xi)\exp(\tfrac{1}{2}\mathrm{i}\sigma^a\lambda_a).\end{aligned}$$

Thus

$$S = \exp(\tfrac{1}{2}\mathrm{i}\sigma^a\lambda_a).$$

For the particular case of $\xi \to '\xi$ belonging to the adjoint group (e.g. pure isospin transformations) σ^a are just equal to η^a_{L}. This follows from the relation

$$'\xi = \eta\xi\eta^{-1}$$

or

$$\exp(-'\xi^a C_a) = \exp(-\eta^a C_a)\exp(-\xi^a C_a)\exp(\eta^a C_a) \qquad (13)$$

and the fact that this relation is a linear one between $'\xi^a$ and ξ^a (see I).

In the case of pure chiral transformations $\xi \to '\xi = \eta\xi\eta$ we shall proceed by determining the infinitesimal transformation. For the moment let us work with a general small change in ξ, viz.

$$\mathrm{d}\xi^\alpha = (A^\alpha_a\,\mathrm{d}\eta^a + A^\alpha_A\,\mathrm{d}\eta^A).$$

The adjoint subgroup results from the restriction $\mathrm{d}\eta^A = -\delta^A_a\,\mathrm{d}\eta^a$ and the chiral subgroup from $\mathrm{d}\eta^A = \delta^A_a\,\mathrm{d}\eta^a$.

We need the relation between $\exp(\tfrac{1}{2}'\xi^a C_a)$ and $\exp(\tfrac{1}{2}\xi^a C_a)$. This is obtained by expansion:

$$\exp(\tfrac{1}{2}'\xi^a C_a) \simeq \partial_\alpha\{\exp(\tfrac{1}{2}\xi^a C_a)\}\,\mathrm{d}\xi^\alpha + \exp(\tfrac{1}{2}\xi^a C_a).$$

We now use the theorem

$$\mathrm{d}\{\exp(M)\} = \exp M\int_0^1 \exp(-tM)\,\mathrm{d}M\exp(tM)\,\mathrm{d}t$$

and the generator property

$$\exp(-\tau^a C_a)C_b\exp(\tau^a C_a) = C_d[\exp(-\tau^a C_a)]^d_{.b} \qquad (14)$$

to write

$$\exp(\tfrac{1}{2}'\xi^a C_a) \simeq \exp(\tfrac{1}{2}\xi^a C_a)\left[1 + \tfrac{1}{2}C_d\int_0^1[\exp(-\tfrac{1}{2}t\xi^a C_a)]^d_{.b}\,\mathrm{d}t\,\partial_\alpha\xi^b\,\mathrm{d}\xi^\alpha\right].$$

† For the time being we retain the indices in, what is, a matrix equation.

Performing the integration we arrive at the result

$$\exp(\tfrac{1}{2}'\xi^a C_a) \simeq \exp(\tfrac{1}{2}\xi^a C_a)\left\{1 - C_d\left[\frac{\exp(-\tfrac{1}{2}\xi^a C_a)-1}{\xi^a C_a}\right]^d_{.b} \mathrm{d}\xi^\alpha \partial_\alpha \xi^b\right\}. \tag{15}$$

For the quantity $\mathrm{d}\xi^\alpha \partial_\alpha \xi^b$ we have

$$\mathrm{d}\xi^\alpha \partial_\alpha \xi^b = \mathrm{d}\xi^b = \delta^b_i(A^i_a \,\mathrm{d}\eta^a + A^i_A \,\mathrm{d}\eta^A)$$

depending only on the A objects in canonical coordinates—indicated by the indices i, j, etc. Let us denote the expression in curly brackets in equation (15) by

$$(1 - \mathrm{d}\tau^d C_d).$$

Equation (12) then yields after substitution from (15) and expansion in η^b_{L} and σ^a, which are assumed small,

$$\exp(\tfrac{1}{2}\xi^a C_a)(\mathrm{d}\eta^b_{\mathrm{L}} + \mathrm{d}\tau^b)C_b \exp(-\tfrac{1}{2}\xi^a C_a) = \mathrm{d}\sigma^d C_d.$$

Whence, using (14),

$$\mathrm{d}\sigma^d = [\exp(\tfrac{1}{2}\xi^a C_a)]^d_{.b}(\mathrm{d}\eta^b_{\mathrm{L}} + \mathrm{d}\tau^b) \tag{16}$$

where

$$\mathrm{d}\tau^b = \left[\frac{\exp(-\tfrac{1}{2}\xi^a C_a)-1}{\xi^a C_a}\right]^b_{.d} \mathrm{d}\xi^d. \tag{17}$$

We can check that this gives the correct result for the adjoint case. Then we have for $\mathrm{d}\xi^d$ the expression (see I, or (13))

$$\mathrm{d}\xi^d = [\xi^a C_a]^d_b \,\mathrm{d}\eta^b_{\mathrm{L}}$$

and it is easy to see that this, together with (16) and (17), implies that $\mathrm{d}\sigma^a$ equals $\mathrm{d}\eta^a_{\mathrm{L}}$, as required.

For the pure chiral transformation, $\mathrm{d}\xi^d$ is given by (see I (19), (25))

$$\mathrm{d}\xi^d \equiv F^d_b \,\mathrm{d}\eta^b_{\mathrm{L}} = [\xi^a C_a \coth(\tfrac{1}{2}\xi^c C_c)]^d_{.b} \,\mathrm{d}\eta^b_{\mathrm{L}}$$

and a little algebra soon yields the answer

$$\mathrm{d}\sigma^d = [\tanh(\tfrac{1}{4}\xi^a C_a)]^d_{.b} \,\mathrm{d}\eta^b_{\mathrm{L}} = v^d_{.b} \,\mathrm{d}\eta^b_{\mathrm{L}}. \tag{18}$$

The $v^d_{.b}$ quantity hereby defined is the same as that introduced by Weinberg (1968, equation (3.1)) and developed by Macfarlane, Sudbery and Weisz (unpublished).

As in I, for the simplest case of chiral SU(2) the expression for $v^d_{.b}$ may be simplified somewhat. We find

$$v^d_{.b} = \left\{\frac{\xi^d \xi_e}{\xi^2}\left(\frac{\tan\frac{1}{4}\xi}{\xi} - \frac{1}{4}\right) - \delta^d_{.e}\frac{\tan\frac{1}{4}\xi}{\xi}\right\}\xi^a \epsilon_{ab}{}^{e}, \qquad \xi = (\xi^a \xi^a)^{1/2} \tag{19}$$

which fits in, of course, with Weinberg's general expression (1968, equation (3.5)).

The corresponding transformations of the $\overset{\mathrm{R}}{\phi}_0$ fields can be written down immediately since we know already that they transform like the $\overset{\mathrm{L}}{\phi}_0$. From their definition,

$$\overset{\mathrm{R}}{\underset{c}{\lambda}}_\alpha = \lambda_A \delta^A_a [\exp(-\tfrac{1}{2}\xi^c C_c)]^a_{.b}\, \delta^b_B A^B_\alpha$$

the $\overset{\mathrm{R}}{\underset{\circ}{\lambda}}_\alpha$ are related to the $\overset{\mathrm{L}}{\underset{\circ}{\lambda}}_\alpha$ by

$$\overset{\mathrm{R}}{\underset{\circ}{\lambda}}_\alpha = -\overset{\mathrm{L}}{\underset{\circ}{\lambda}}_\alpha$$

using (11).

The restriction of the present discussion to quark fields is not a serious one and can easily be lifted. If one does not wish to talk about fields transforming one must introduce quantities analogous to the $t^{(\mu)}$ objects of Lorentz group theory (see e.g. Dowker and Goldstone 1968, Dowker and Dowker 1966, where the original references are given).

The answer is that for the general zero fields $\overset{\mathrm{L}}{\phi}_0$ the transformation matrix S is given, infinitesimally, by

$$S = 1 + \mathrm{i}\,\mathrm{d}\sigma^d J_d$$

where, for the adjoint case, $\mathrm{d}\sigma^d = \mathrm{d}\eta^d$ and for the pure chiral case $\mathrm{d}\sigma^d$ is given by (18). Similar results hold for the right fields $\overset{\mathrm{R}}{\phi}_0$.

If we want to write down a Lagrangian including quark fields then naturally we shall need derivatives and, as in general relativity, the concept of covariant derivative arises. We thus turn to this topic.

3. Covariant derivatives

We refer to our paper (Dowker and Dowker 1966) and to references therein for a relevant discussion of the covariant derivatives of spin functions in curved space. We shall review the situation here for the case in point.

The formalism is covariant under the group of 'spin' basis changes, i.e. the transformations

$$\phi \to S\phi, \qquad \lambda^\alpha \to (S^\dagger)^{-1}\lambda^\alpha S^{-1}$$

where S is an *arbitrary* matrix, constitute merely a change of description exactly analogous to general coordinate transformations. We should thus like to write the theory in a form manifestly covariant under these transformations, hence a covariant derivative for 'spinor' quantities.

The covariant derivative of ϕ is defined by

$$\nabla_\beta\phi = \partial_\beta\phi + \Gamma_\beta\phi$$

where Γ_β is a matrix in 'quark' space (what we have called 'spin' space previously).

Now the quantity $\phi^\dagger\lambda^\alpha\phi$ is a vector field and so, by taking its covariant derivative, that of λ^α can be found. Thus

$$\nabla_\beta(\phi^\dagger\lambda_\alpha\phi) = (\phi^\dagger\lambda_\alpha\phi)_{\|\beta} \tag{20}$$

where $\|$ indicates the tensor covariant derivative with respect to the 'zero' connection, i.e. the usual covariant derivative in a Riemannian space (see I). If we expand (20) using the distributive rule we find

$$\nabla_\beta\lambda_\alpha = \lambda_{\alpha\|\beta} - \Gamma_\beta{}^\dagger\lambda_\alpha - \lambda_\alpha\Gamma_\beta.$$

Although it is not necessary, we are going to choose that $\nabla_\beta\lambda_\alpha$ vanish. This is usually done in the analogous spin-in-curved-space problem. Thus we have

$$\lambda_{\alpha\|\beta} = \Gamma_\beta^{\,\dagger}\lambda_\alpha + \lambda_\alpha\Gamma_\beta \tag{21}$$

from which equation an explicit form for Γ_β can be found if we assume the general form

$$\Gamma_\beta = \mathrm{i}\Omega^{\alpha}_{.\beta}\lambda_\alpha \qquad \Omega \text{ real.} \tag{22}$$

Equations (21) and (22) yield, if the commutation rules for the λ_α are employed,

$$\pm\lambda_\delta c^{\cdot\cdot\delta}_{\alpha\gamma}\,\Omega^{\gamma}_{\cdot\beta} = -\tfrac{1}{2}\lambda_{\alpha\|\beta} \qquad \begin{pmatrix} + \text{ for L} \\ - \text{ for R} \end{pmatrix}. \tag{23}$$

To proceed further we need an expression for $\lambda_{\alpha\|\beta}$ as a linear combination of λ_δ. This will follow if we give the relation between λ_α and λ_a (or λ_A for right fields). We are really only interested in two such relations: those which define the $\overset{L}{\phi}$ and $\overset{L}{\phi}_0$ fields†, i.e.

$$\lambda_\alpha = \overset{L}{\lambda}_\alpha = \lambda_a A^a_\alpha \tag{24}$$

$$\lambda_\alpha = \underset{o}{\overset{L}{\lambda}}_\alpha = \lambda_d[\exp(\tfrac{1}{2}\xi^a C_a)]^d_{\cdot b}A^b_\alpha. \tag{25}$$

Equations (23) and (24) easily give the result for the $\overset{L}{\phi}$ fields

$$\nabla_\beta\overset{L}{\phi} = (\partial_\beta - \tfrac{1}{4}i\overset{L}{\lambda}_\beta)\overset{L}{\phi}, \qquad \overset{L}{\Omega}{}^{\alpha}_{\cdot\beta} = -\tfrac{1}{4}\delta^\alpha_\beta. \tag{26}$$

The calculation is a little more involved for $\overset{L}{\phi}_0$ and we again encounter the derivative $\partial_\beta\{\exp(\frac{1}{2}\xi^a C_a)\}$ calculated in the last paragraph. We find finally

$$\underset{o}{\overset{L}{\Omega}}{}^{\alpha}_{\cdot\beta} = \tfrac{1}{4}A^\alpha_a[\tanh(\tfrac{1}{4}\xi^c C_c)]^a_{\cdot b}A^b_\beta. \tag{27}$$

If we use canonical coordinates for the ξ we have

$$\underset{o}{\overset{L}{\Omega}}{}^{i}_{\cdot j} = \tfrac{1}{4}\delta^i_a[\tanh(\tfrac{1}{4}\xi^c C_c)]^a_{\cdot b}\delta^b_j. \tag{28}$$

Expressions (26) and (27) could have been derived directly from the results of § 2 (cf. equation (18)).

Similar results hold for the right fields $\overset{R}{\phi}$ and $\overset{R}{\phi}_0$. We have

$$\overset{R}{\Omega}{}^{\alpha}_{\cdot\beta} = -\tfrac{1}{4}\delta^\alpha_\beta$$

and

$$\underset{o}{\overset{R}{\Omega}}{}^{\alpha}_{\cdot\beta} = -\tfrac{1}{4}A^\alpha_A\delta^A_a[\tanh(\tfrac{1}{4}\xi^c C_c)]^a_{\cdot b}\delta^b_B A^B_\beta.$$

The ∇_α derivatives are, of course, covariant derivatives in group space. For insertion in a Lagrangian we need space–time derivatives. With Meetz (1969) we define the space–time covariant derivative of a quantity Φ by

$$\begin{aligned} D_\mu\Phi &= \partial_\mu\Phi + \partial_\mu\xi^\alpha\nabla_\alpha\Phi, \qquad \partial_\mu = \frac{\partial}{\partial x^\mu} \\ &= \Phi_{,\mu} + \Gamma_\mu\Phi \end{aligned} \tag{29}$$

where

$$\Phi_{,\mu} \equiv \partial_\mu\Phi + \partial_\mu\xi^\alpha\partial_\alpha\Phi$$

$$\Gamma_\mu \equiv \partial_\mu\xi^\alpha\Gamma_\alpha.$$

Space–time tensor indices are taken from the middle of the Greek alphabet, μ, ν, etc. Since the space–time point x^μ is a parameter in all quantities of the preceding analysis, Φ will be a function of x through ξ and also explicitly, in general. The symbol ∂_μ acts only on the explicitly occurring x's.

† The corresponding results for the right fields need only be written down.

In terms of the constant λ_a, we find for $\overset{\mathrm{L}}{\Gamma}_\mu$ and $\overset{\mathrm{L}}{\underset{\circ}{\Gamma}}_\mu$ the expressions

$$\overset{\mathrm{L}}{\Gamma}_\mu = -\tfrac{1}{4}\mathrm{i}\tilde{\lambda}\exp(-V)\frac{\sinh V}{V}\,\partial_\mu\xi \equiv -\tfrac{1}{4}\mathrm{i}\tilde{\lambda}\exp(-V)D_\mu\xi$$

$$\overset{\mathrm{L}}{\underset{\circ}{\Gamma}}_\mu = \tfrac{1}{4}\mathrm{i}\tilde{\lambda}\frac{\cosh V-1}{V}\,\partial_\mu\xi = \tfrac{1}{4}\mathrm{i}\tilde{\lambda}\tanh\left(\frac{V}{2}\right)D_\mu\xi \tag{30}$$

where $V = \tfrac{1}{2}U = \tfrac{1}{2}C_a\xi^a$. For compactness we have now gone over into a matrix notation. Thus quantities with superscript indices, for example ξ^a, are column matrices ξ, while those with subscript ones, for example λ_a, are row matrices $\tilde{\lambda}$.

As matrices, in quark space, $\overset{\mathrm{R}}{\Gamma}_\mu$ and $\overset{\mathrm{R}}{\underset{\circ}{\Gamma}}_\mu$ are given by

$$\overset{\mathrm{R}}{\Gamma}_\mu(\xi^i) = \overset{\mathrm{L}}{\Gamma}_\mu(-\xi^i)$$

$$\overset{\mathrm{R}}{\underset{\circ}{\Gamma}}_\mu = \overset{\mathrm{L}}{\underset{\circ}{\Gamma}}_\mu. \tag{31}$$

Expression (30) is identical with that of Callan *et al.* (1969) if we note that their ξ is equal to half the ξ used in this paper. The construction of specific Lagrangians can now proceed along the lines indicated in their paper. Thus we would have the quark Lagrangian, with 'minimal' meson interactions

$$\mathscr{L} = \mathrm{i}\left\{\overset{\mathrm{L}\dagger}{\varphi}_0\sigma^\mu \overset{\mathrm{L}}{D}_\mu\varphi_0 + \overset{\mathrm{R}\dagger}{\chi}_0\bar{\sigma}^\mu \overset{\mathrm{R}}{D}_\mu\chi_0 - \kappa\left(\overset{\mathrm{R}\dagger}{\chi}_0\overset{\mathrm{L}}{\varphi}_0 - \overset{\mathrm{L}\dagger}{\varphi}_0\overset{\mathrm{R}}{\chi}_0\right)\right\}. \tag{32}$$

The particular meson–quark coupling implied by this Lagrangian is, at least to lowest order in ξ, of the 'current–current' type, for example for SU(2)

$$\bar{\psi}_0\gamma^\mu\boldsymbol{\tau}\psi_0\,.\,(\boldsymbol{\pi}_\wedge\partial_\mu\boldsymbol{\pi}), \qquad \psi_0 = \begin{pmatrix}\overset{\mathrm{L}}{\varphi}_0\\ \overset{\mathrm{R}}{\chi}_0\end{pmatrix}.$$

If we wish to include an ordinary pseudovector coupling then this must be added in non-minimally. It is contained in the term

$$\bar{\psi}_0\gamma^\mu\gamma^5\underset{\circ}{\lambda}_\alpha\psi_0\partial_\mu\xi^\alpha = \bar{\psi}_0\gamma^\mu\gamma^5\tilde{\lambda}\psi_0\exp(V)A_\alpha\partial_\mu\xi^\alpha = \bar{\psi}_0\gamma^\mu\gamma^5\tilde{\lambda}\psi_0\,.\,D_\mu\xi \tag{33}$$

$$D_\mu\xi \equiv \frac{\sinh V}{V}\,\partial_\mu\xi. \tag{33}$$

$D_\mu\xi$ is the meson 'covariant derivative' of Weinberg (1968) and Callan *et al.* (1969). Its appearance is simply due to the fact that we have reintroduced the constant matrices λ_a. In the present approach $D_\mu\xi$ is not a fundamental object but appears when we write the theory in terms of tangent space quantities.

From this result follows the theorem of Weinberg (1968) and Colman *et al.* (1969) that a Lagrangian which is SU(n) invariant and is constructed from ϕ_0, $D_\mu\phi_0$ and $D_\mu\xi$ is also chiral invariant.

It is only necessary to replace $\tfrac{1}{2}\lambda_a$ by J_a in the above to obtain the corresponding results for representations higher than the quark one.

At this point we shall end the didactic part of this paper and shall pass on to some (speculative) comments.

4. Comments

The introduction of a non-integrable connection related to the meson fields allows us to discuss all the topics associated with a gauge-field theory. Thus a generalized Aharonov–Bohm situation can be visualized, if not actually realized. The analogues of Dirac monopoles can be discussed and here the (known) topology of group space would be significant (the first, and last, two Betti numbers vanish, etc.: for example, see Hodge 1952), and so on.

We should, of course, remember that the physical interaction may not be purely minimal and in this connection we note that the term of equation (33) could be included in (32) by replacing D_μ by $\overset{*}{D}_\mu$ where

$$\overset{*}{D}_\mu\psi \equiv D_\mu\psi + g\tilde{\lambda}D_\mu\xi\gamma^5\psi, \qquad g = \text{constant}.$$

Another possible problem is associated with the non-commutativity of the ∇_α and hence of the D_μ. Thus we find

$$D_{[\mu}D_{\nu]}\phi = -\tfrac{1}{2}\mathrm{i}R_{\mu\nu}\phi, \qquad R_{\mu\nu} = \tfrac{1}{4}\partial_\mu\xi^\alpha\partial_\nu\xi^\beta c_{\alpha\beta}^{\cdot\cdot\,\gamma}\lambda_\gamma$$

where $R_{\mu\nu}$ is a sort of 'spin curvature'. If we want to write down interacting field equations for particles of higher spin then consistency problems will arise (e.g. Fierz and Pauli 1939). It may be that these considerations are nugatory if we take the S-matrix approach.

5. Conclusion

We have seen that the 'other fields' are to be associated with the group of local rotations in group space and that different choices of the corresponding local frames give different fields, i.e. fields with different transformations. If these local frames are chosen to coincide with the A_α^a or A_α^A vectors, which determine the structure of the space, then the fields so obtained transform linearly. If we take the frames obtained by rotating the A_α^a and A_α^A half way towards each other then we obtain two sets of fields transforming non-linearly but cogrediently. This rotation to symmetrically placed local axes is the Gürsey transformation. It should be clear that we will obtain fields which transform cogrediently if we choose coincident local frames, i.e. frames obtained by rotating A_α^a and A_α^A until they overlap.

For chiral SU(2) any fields but pions will be 'other fields'. Thus, for example, the kaons will be introduced through local, i.e. tangent space, considerations. On the other hand, in chiral SU(3) the kaons will in fact form part of the octet of coordinatizing fields. No doubt this sort of situation could be described geometrically in terms of the embedding of the corresponding group spaces.

It is possible to question the logical position of the fields forming the coordinates of group space. If nuclear democracy means anything then all fields should be on the same footing, i.e. all physical fields will be 'other fields'. There is also nothing to stop us introducing the spaces associated with *all* the representations of G and not just the adjoint one. These spaces are more complicated since G does not, in general, act transitively on them.

References

Biedenharn, L. C., 1963, *Lectures in Theoretical Physics*, Vol. 5 (New York: Interscience).
Brown, L. S., 1967, *Phys. Rev.*, **163**, 1802–7.
Callan, C. G., Coleman, S., Wess, J., and Zumino, B., 1969, *Phys. Rev.*, **177**, 2247–50.
Cartan, E., 1938, *Théorie des Spineurs* (Paris: Hermann).

Chang, P., and Gürsey, F., 1967, *Phys. Rev.*, **164,** 1752–61.
Coleman, S., Wess, J., and Zumino, B., 1969, *Phys. Rev.*, **177,** 2239–47.
Cronin, J. A., 1967, *Phys. Rev.*, **161,** 1483–94.
De Witt, B. S., and De Witt, C. M., 1952, *Phys. Rev.*, **87,** 116–22.
Dowker, J. S., 1970, *J. Phys.* A: *Gen. Phys.*, **3,** 33–44.
Dowker, J. S., and Dowker, Y. P., 1966, *Proc. Phys. Soc.*, **87,** 65–78.
Dowker, J. S., and Goldstone, M., 1968, *Proc. R. Soc.* A, **303,** 381–96.
Fierz, M., and Pauli, W., 1939, *Proc. R. Soc.* A, **173,** 211–32.
Gürsey, F., 1960, *Nuovo Cim.*, **16,** 230–40.
Hodge, W. V. D., 1952, *Theory and Applications of Harmonic Integrals* (London: Cambridge University Press).
Meetz, K., 1969, *J. Math. Phys.*, **10,** 589–93.
Schouten, J. A., 1954, *Ricci-Calculus* (Berlin: Springer-Verlag).
Weinberg, S., 1968, *Phys. Rev.*, **166,** 1568–77.

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